Generators of Jacobian Groups of Graphs

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What are Graphs?

Definition (Graph)

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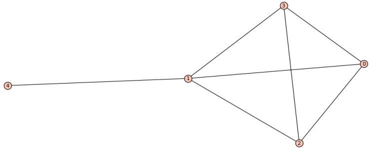

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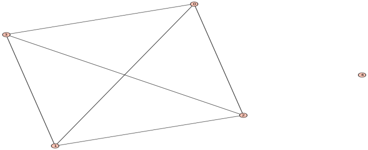

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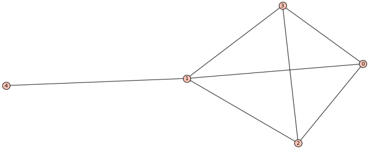

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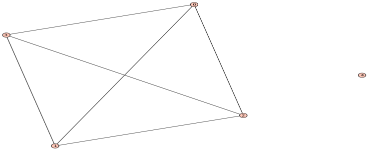
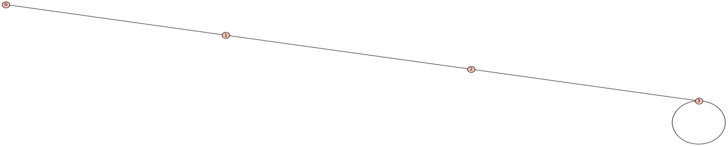


Graph with loops:



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1. To study geometric objects, useful to assign algebraic structures to them
2. Eg. symmetry groups of shapes, groups on elliptic curves, etc.
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2. Eg. symmetry groups of shapes, groups on elliptic curves, etc.
3. Inspired by that, we do the same for graphs
4. The elements of these groups will be called divisors

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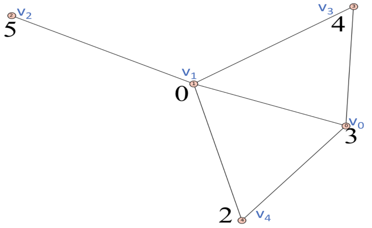

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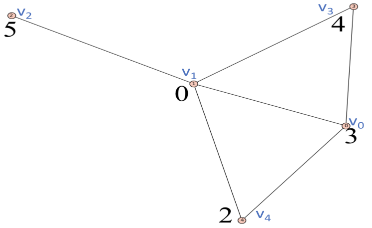

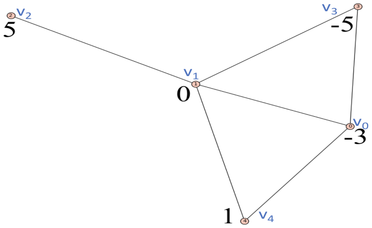

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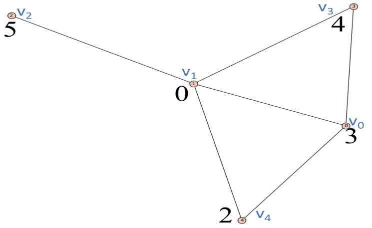

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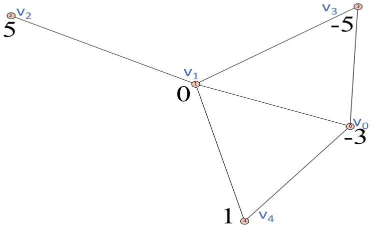

We can add the divisors as follows:

$D_1 + D_2 = 0v_0 + 0v_1 + 10v_2 - 1v_3 + 3v_4$.

Divisors

Theorem

The set of divisors of a graph G form an Abelian group $\mathrm{Div}(G)$ under this operation.

There is an identity, inverses exist, the operation is closed, associative, and commutative.

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The set of divisors with degree 0 form a subgroup $\mathrm{Div}_0(G)$ of $\mathrm{Div}(G)$.

Sum of degree zero divisors must have degree zero. Inverses of degree zero divisors must have degree zero.

Chip Firing

- Think of a divisor as assigning a number of "chips" to each node
- When a node "fires," it sends one chip along each edge
- Note that this operation preserves the degree of the divisor

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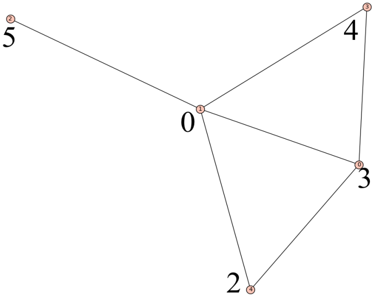

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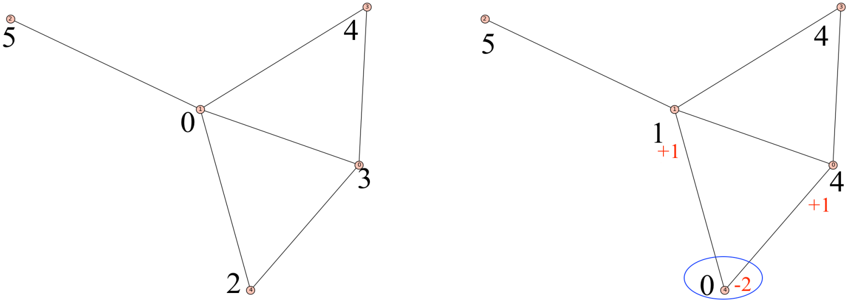

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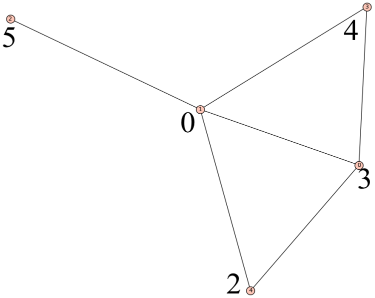

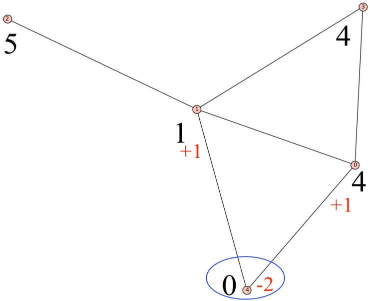

Definition (Firing Script)

A firing script σ is an integer vector $\sigma \in \mathbb{Z}^n$ whose entries specify the number of times each node of a divisor should be fired.

Chip Firing as an Equivalence Relation

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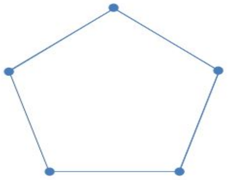
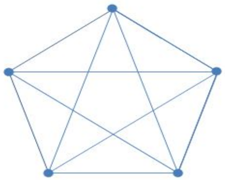
Definition (Jacobian Group)

Let G be a graph. The Jacobian group Jac(G) is defined as $\mathrm{Div}_0(G)/\,\mathrm{Prin(G)}$.

Examples of Jacobians





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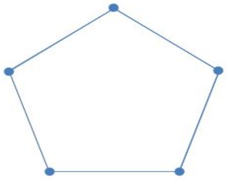

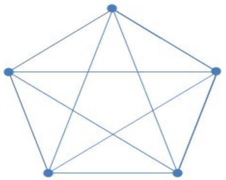

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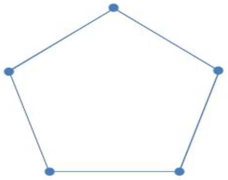

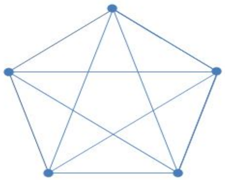

1. The linear graph L_n has a trivial Jacobian.
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3. The complete graph K_n has $\mathrm{Jac(K_n)} = \mathbb{Z}_n^{n-2}$

The Laplacian

Definition

Let G be a graph. Let Δ be a diagonal matrix where $\Delta_{(i,i)}$ equals the number of edges incident to vertex i. Let A be the adjacency matrix of G. Then the Laplacian $L := \Delta - A$.

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2. If we take the zero-divisor and fire the nodes by σ, the resulting divisor is $D = L\sigma$.
3. $\det(L) = |\operatorname{Jac}(G)|$.
4. The entries of the SNF of the Laplacian are the invariants of Jac(G).

Previous Results

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2. Lorenzini determined effect of adding or removing an edge of G on the size of its minimal generating sets

Theorem (Lorenzini 1989)

Let G be a connected graph. Let G' be a connected graph formed by removing an edge of G. Then the size of the minimal generating set of $\mathrm{Jac}(G')$ differs from that of $\mathrm{Jac}(G)$ by at most 1.

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Theorem (Brandfonbrener et. al. 2017)

Let G be a graph and G_1 be a connected graph formed by adding/removing the edge between x and y. The divisor δ_{xy} is a generator of $\mathrm{Jac}(G)$ *if and only if*

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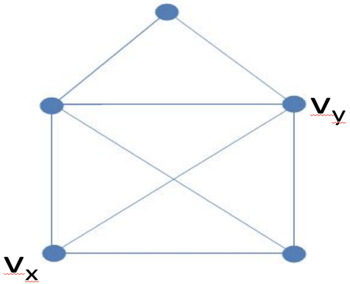
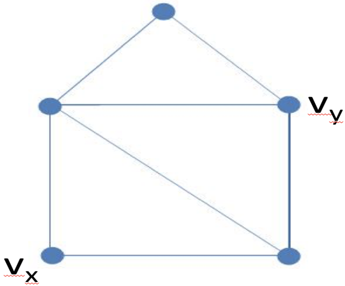
G with $|\,\mathrm{Jac}(G)| = 40$:



G_1 with $|\,\mathrm{Jac}(G_1)| = 21$:



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2. Working with δ-divisors is a place to start
3. We developed a procedure which we conjecture produces a smallest generating set consisting of only δ-divisors

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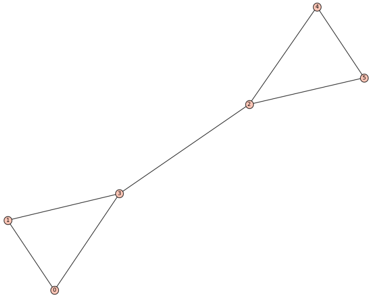
Conjecture

If the above procedure terminates, then $\langle \delta_{x_1 y_1}, \ldots, \delta_{x_n y_n} \rangle = \mathrm{Jac}(G)$.

New Procedure Example



1. $|\operatorname{Jac}(G)| = 9$. The largest subgroups generated by a δ_{xy} have order 3. Set δ_{xy1} equal to one such divisor, for example δ_{01}.

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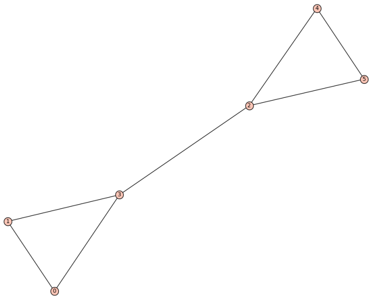

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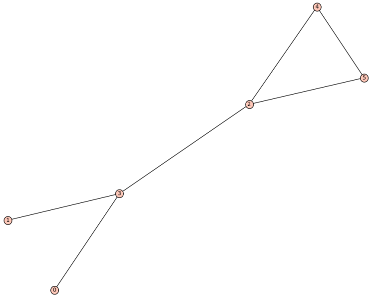

1. $|\operatorname{Jac}(G)| = 9$. The largest subgroups generated by a δ_{xy} have order 3. Set δ_{xy1} equal to one such divisor, for example δ_{01}.
2. G contains a $0-1$ edge, so remove it to form G_1.
3. $|\operatorname{Jac}(G_1)| = 3$. The largest subgroups generated by a δ_{xy} have order 3. Set δ_{xy2} equal to one such divisor, for example δ_{45}.

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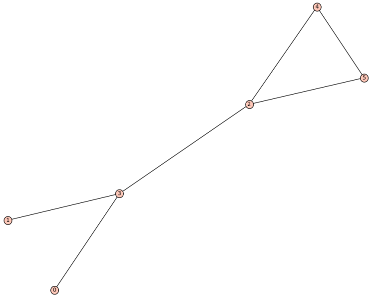

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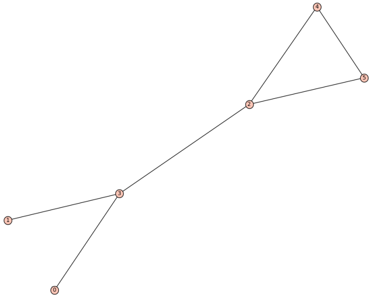

Example

Computationally, we can check that $\langle \delta_{01}, \delta_{45} \rangle = \mathrm{Jac}(G)$.

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Computational Support

1. Wrote software to apply procedure to randomly generated graphs.
2. About 1000 across graphs of $4 - 10$ nodes.
3. In 99 percent of trials, the process terminated.
4. All terminated trials resulted in a generating set for the original graph.

Future Research

1. Prove that when the procedure terminates, it produces a generating set for the Jacobian of the original graph.
2. With what probability does the procedure terminate?
3. With what probability does it produce a generating set with minimal order

Acknowledgements

I'd like to thank

1. My mentor, Dr. Xiaomeng Xu
2. Dr. Dhruv Ranganathan
3. The PRIMES program and MIT Math Department
4. Dr. Tanya Khovanova
5. My parents